Symmetry groups and non-planar collisionless action-minimizing solutions of the three-body problem in three-dimensional space

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Abstract

Periodic and quasi-periodic solutions of the n-body problem can be found as minimizers of the Lagrangian action functional restricted to suitable spaces of symmetric paths. The main purpose of this paper is to develop a systematic approach to the equivariant minimization for the three-body problem in the three-dimensional space. First we give a finite complete list of symmetry groups fit to the minimization of the action, with the property that any other symmetry group can be reduced to be isomorphic to one of these representatives. A second step is to prove that the resulting (local and global) symmetric action-minimizers are always collisionless (when they are not already bound to collisions). Furthermore, we prove some results addressed to the question whether minimizers are planar or non-planar; as a consequence of the theory we will give general criteria for a symmetry group to yield planar or homographic minimizers (either homographic or not, as in the Chenciner-Montgomery eight solution); on the other hand we will provide a rigorous proof of the existence of some interesting one-parameter families of periodic and quasi-periodic non-planar orbits. These include the choreographic Marchal's P_{12} family with equal masses – together with a less-symmetric choreographic family (which anyway probably coincides with the P_{12}).

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1 Introduction

In some recent papers classical variational methods have been successfully applied in the proof of the existence of periodic or quasi-periodic solutions for the n-body prob-

lem. Suitable symmetry groups of the Lagrangian action functional have been introduced and exploited in order to apply the aforementioned techniques to the class of all symmetric loops, to name a few, in the articles [2, 3, 4, 10, 11, 12, 15, 18, 19]. Surveys and further details on this approach can be found for example in [1, 5, 7, 12, 17]. The major problem in the search of equivariant minimizers is that of collisions: a (local or global) minimizer might consist *a priori* of a colliding trajectory. The latest significant breakthrough in this direction has been allowed by Marchal's averaging technique [5, 12]. In the paper [12] the authors develop a general theory for G-equivariant minimizers and present a class of groups that yield always, as a consequence of Marchal's averaging technique, collision-free minimizers. In [1] this result was extended to all possible symmetry groups for the planar three-body problem. A naturally related problem is to find and classify all possible symmetry groups and to understand whether the resulting minima are rotating central configurations or if they are new solutions (and, at the same time, to provide rigorous proofs of the existence and of the properties of some solutions whose existence was accepted as a fact after numerical evidence). This has been done for the planar problem by V. Barutello, S. Terracini and the author in [1]. The purpose of the paper is to give a complete answer to the classification problem for the three-body problem in the space and at the same time to determine and describe properties of the resulting minimizers. In particular, we focus on non-planar orbits, since planar orbits have been already included in the list of [1]. In order to state the main results, we anticipatively sketch some basic definitions: A symmetry group G of the Lagrangian functional $\mathcal{A}$ (see (2.5) below) is termed *bound to collisions* if all G-equivariant loops actually have collisions (see (2.6) below), *fully uncoercive* if for every possible rotation vector $\underline{\omega}$ the action functional $\mathcal{A}^G_{\underline{\omega}}$ in the frame rotating around $\underline{\omega}$ with angular speed $|\underline{\omega}|$ is not coercive in the space of G-equivariant loops (that is, its global minimum escapes to infinity – see (2.17)); moreover, G is termed *homographic* if all G-equivariant loops are constant up to orthogonal motions and rescaling. Note that if there is a rotation axis $\underline{\omega}$ then the group G is implicitly assumed to be a symmetry group of the action functional $\mathcal{A}_{\underline{\omega}}$ in the rotating frame (that is, the functional including the centrifuge and Coriolis terms); such a group is termed *of type R* (see (2.14) below); finally, the *core* of the group G is the subgroup of all the elements which do not move the time $t \in \mathbb{T}$ (see (2.8) below). In the first theorem we classify symmetry groups, up to change in rotating frame. For the symbols used we refer to sections 2 and 3 below.

Theorem A. *Symmetry groups not bound to collisions, not fully uncoercive and not homographic are, up to a change of rotating frame, either the three-dimensional extensions of planar groups (if trivial core) listed in table 1 or the vertical isosceles triangle (6.2) (if non-trivial core).*

The next theorem is the answer to the natural questions about collisions and description of some main features of minimizers.

Theorem B. *Let G be a symmetry group not bound to collisions and not fully uncoercive. Then*

(i) *Local minima of $\mathcal{A}^G_{\underline{\omega}}$ do not have collisions.*

(ii) *In the following cases minimizers are planar trajectories:*

2

Table 1: Space extensions of planar symmetry groups with trivial core

Name	Extensions
Trivial	C_1^-
Line	$L_2^{+,-}$, $L_2^{-,+}$
Isosceles	$H_2^{+,-}$, $H_2^{-,+}$
Hill	$H_4^{+,-}$, $H_4^{-,+}$
3-choreography	C_3^+, C_3^-
Lagrange	$L_6^{+,+}$, $L_6^{+,-}$, $L_6^{-,+}$
D_6	$D_6^{+,-}$, $D_6^{-,+}$
D_{12}	$D_{12}^{-,+}$

(a) If G is not of type R: $D_6^{+,-}$, $D_6^{-,+}$ and $D_{12}^{-,+}$ (and then G-equivariant minimizers are Chenciner–Montgomery eights).

(b) If there is a G-equivariant minimal Lagrange rotating solution: C_1^-, $H_2^{+,-}$, C_3^+, $L_6^{+,+}$, $L_6^{+,-}$ (and then the Lagrange solution is of course the minimizer).

(c) If the core is non-trivial and it is not the vertical isosceles (6.2) (and then minimizers are homographic).

(iii) In the following cases minimizers are always non-planar:

(a) The groups $L_6^{-,+}$ and C_3^- for all $\omega \in (-1, 1) + 6\mathbb{Z}$, $\omega \neq 0$ (the minimizers for $L_6^{-,+}$ are the elements of Marchal family P_{12}, and minimizers of C_3^- are a less-symmetric family P'_{12} [1]).

(b) The extensions of line and Hill-Euler type groups, for on open subset of mass distributions and angular speeds ω (explicitely given in (5.8)): $L_2^{+,-}$, $L_2^{-,+}$, $H_4^{+,-}$ and $H_4^{-,+}$ (for $L_2^{-,+}$ this happens also with equal masses).

(c) The vertical isosceles (6.2) for suitable choices of masses and ω.

In the article we develop the needed tools and prove these statements, after the necessary explanations about preliminary results and notation. Together with the results of [12] and [1] it is exposed a theory of action-minimizing symmetry periodic n-body orbits. In section 2 we introduce all the definitions needed in the sequel and prove some preliminary results. In section 3 we introduce the concept of three-dimensional *extension* of a planar symmetry group so that we can use the classification of [1]. In section 4 the angular momentum J enters into account, and we show how the existence of rotation axes is related to the possibility of being non-zero of J. Afterward, in section 5 we prove some interesting estimates on second variations, which are remarkably simple (incidentally, they work not only for 3 bodies, but for n arbitrary). It is by an application of these simple estimates that one proves the fact that the non-planar quasi-periodic orbits listed in theorem B exist. In

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[1]Highly likely they are not distinct families: this is the recurring phenomenon of "more symmetries than expected" in n-body problems.

section 6 we come to the classification of three-dimensional space symmetries, which is a proof of theorem A. The proof of the various items of theorem B is done in section 7. Finally, in section 8 some concluding remarks are collected.

Before we start with the next section, a few words have to be spent on the existence of the P_{12}. A different – and very elementar – proof of the existence of the P_{12}-family with D_{12}-symmetries was presented by A. Chenciner in [5, 7], which does not require local results on collisions, since collisions are excluded by action level estimates. The advantage of our approach is that it can be plainly extended to the case of any odd number $n \geq 3$ of bodies in the space (see remark (8.5) below). All other results are, to our knowledge, new: whenever similar methods or results were published elsewhere, it has been remarked in-place.

2   Preliminaries

Consider the linear space of configurations with center of mass in 0

$$\mathcal{X} = \{x = (x_1, x_2, x_3) \in E^3 \mid m_1 x_1 + m_2 x_2 + m_3 x_3 = 0\}.$$

Let $\mathbb{T} = S^1 \cong \mathbb{R}/2\pi\mathbb{Z}$ be the unit circle of length 2π. We are dealing with periodic orbits of the Newtonian n-body problem, which will be seen as critical points of a suitable functional on the Sobolev space $\Lambda = H^1(\mathbb{T}, \mathcal{X})$ consisting of all L^2 loops $\mathbb{T} \to \mathcal{X}$ with L^2 derivative. It is an Hilbert space with the scalar product

$$x \cdot y = \int_{\mathbb{T}} (x(t)y(t) + \dot{x}(t)\dot{y}(t))dt.$$

The α-homogeneous Newtonian potential can be written as

(2.1)
$$U(x) = \frac{m_1 m_2}{|x_1 - x_2|^\alpha} + \frac{m_1 m_3}{|x_1 - x_3|^\alpha} + \frac{m_2 m_3}{|x_2 - x_3|^\alpha}.$$

Let $\underline{\omega} \in E \cong \mathbb{R}^3$ be a vector. The kinetic form in a frame uniformly rotating around $\underline{\omega}$ with angular speed $\omega = |\underline{\omega}|$ is defined by

(2.2)
$$2K(x, \dot{x}) = \sum_{i=1}^{3} m_i |\dot{x}_i + \Omega x_i|^2,$$

where Ω is the matrix

$$\begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix}$$

obtained by the coefficients $(\omega_1, \omega_2, \omega_3)$ of $\underline{\omega} \in \mathbb{R}^3$. Thus, for every $\underline{\omega}$ the Lagrangian can be written as

(2.3)
$$L_{\underline{\omega}}(x, \dot{x}) = L_{\underline{\omega}} = K_{\underline{\omega}} + U,$$

and, finally, the action functional as

(2.4)
$$\mathcal{A}_{\underline{\omega}}(x) = \int_{\mathbb{T}} L_{\underline{\omega}}(x(t), \dot{x}(t))dt.$$

4

(2.5) Definition. We term *symmetry group* every subgroup of $O(\mathbb{T}) \times O(3) \times \Sigma_3$, where $O(\mathbb{T}) = O(2)$ is the orthogonal group of dimension 2 acting on the time circle, $O(3)$ the orthogonal group of dimension 3 acting on the space E and Σ_3 the symmetric group on the three elements $\{1, 2, 3\}$.

Given a subgroup $G \subset O(\mathbb{T}) \times O(3) \times \Sigma_3$, it is possible to define three homomorphisms $\tau \colon G \to O(\mathbb{T})$, $\rho \colon G \to O(3)$ and $\sigma \colon G \to \Sigma_3$ by projection onto the first, second or third factor of the direct product. Given τ, ρ and σ one can define in an obvious way a G-action on $\mathbb{T}$, E and $\{1, 2, 3\}$, and hence an action on the centered configuration space $\mathcal{X}$, provided that for every $g \in G$ it happens that $m_i = m_{gi}$ (that is, masses are constant in G-orbits in the index set $\{1, 2, 3\}$. Thus, there is an induced action of G on the Sobolev space Λ of loops, defined by $g(x(t)) = (gx)(g^{-1}t)$. The action is orthogonal on Λ so that Palais theorem says that, for a given group G, if the functional $\mathcal{A}_\omega \colon \Lambda \to \mathbb{R}$ is G-invariant and a G-equivariant loop $x(t)$ is collisionless and critical for the restriction $\mathcal{A}_\omega^G = \mathcal{A}_\omega | \Lambda^G$, then $x(t)$ is critical for $\mathcal{A}_\omega$.

(2.6) Definition. A group is termed *bound to collisions* if for every equivariant loop $x(t) \in \Lambda^G$ collisions occur, that is, for each G-equivariant $x(t)$ there is $t_c \in \mathbb{T}$ and $i \neq j \in \mathbf{n} = \{1, 2, 3\}$ such that $x_i(t_c) = x_j(t_c)$.

(2.7) Definition. A group G is termed *homographic* if every equivariant loop $x(t) \in \Lambda^G$ is constant up to rescaling and orthogonal motions.

(2.8) Definition. The kernel $\ker \tau$ is termed the *core* of the symmetry group G.

(2.9) Definition. A group G is termed of *cyclic*, *brake* or *dihedral* type respectively if $G/\ker \tau$ acts orientation-preserving on the time circle $\mathbb{T}$, if $G/\ker \tau$ has order 2 and acts orientation-reversing on $\mathbb{T}$ or if $G/\ker \tau$ is a dihedral group of order ≥ 4.

Consider the following elements in $O(2)$: 1 is the trivial motion, -1 is the rotation of angle π and l is a reflection along a line. Elements of the symmetric group Σ_3 will be denoted in the cyclic permutation notion. Let $\ker \det(\tau) \subset G$ denote the subgroup of G of the elements acting orientation-preserving on $\mathbb{T}$. A symmetry group with trivial core will be fully determined once the images $\rho(r)$ and $\sigma(r)$ of a generator r of $\ker \det(\tau)$ rotating a minimal angle are given, together, if it is not of cyclic type, with the images $\rho(h)$ and $\sigma(h)$ of one of the elements not in $\ker \det(\tau)$ (which have order 2). The full list of representatives of the planar classification exposed in [1] can be therefore found in table 2, with the corresponding generators.

(2.10) Definition. A planar symmetry group G is said *of type R* if the determinant homomorphisms $\det(\rho), \det(\tau) \colon G \to \{+1, -1\}$ coincide, that is, if they coincide as G-representations.

(2.11) Definition. A vector $v \in E \cong \mathbb{R}^3$ is termed a *rotation axis* with respect to a symmetry group G if the line spanned by v in E is G-invariant and the following equality of one-dimensional G-representations holds:

$$\det(\tau)\det(\rho) = \det(v),$$

where $\det(v)$ denotes the real representation of G induced by restricting ρ to the invariant subspace generated by $v \in E$.

Table 2: Planar symmetry groups with trivial core

Name	Symbol	$\rho(r),\sigma(r)$	$\rho(h),\sigma(h)$
Trivial	C_1	1,()	
Line	L_2	1,()	l,()
2-1-choreography	C_2	1, (1,2)	
Isosceles	H_2	1,()	l,(1,2)
Hill	H_4	1,(1,2)	l,(1,2)
3-choreography	C_3	1,(1,2,3)	
Lagrange	L_6	1,(1,2,3)	l,(1,2)
C_6	C_6	l,(1,2,3)	
D_6	D_6	1,(1,2,3)	-1,(1,2)
D_{12}	D_{12} l,(1,2,3)	-1,(1,2)	

(2.12) *The restriction of a three-dimensional symmetry group G to the orthogonal complement of a rotation axis $v \in E$ is a planar symmetry group of type R. Conversely, if the restriction of G to an invariant plane is of type R, then the orthogonal complement of the invariant plane is a rotation axis for G.*

Proof. Let τ, ρ and σ be the defining homomorphisms of G, where $\rho \colon G \to O(3)$ can be written as $\rho = \rho_2 \times \rho_1$, with $\rho_2 \colon G \to O(2)$ induced by restriction to the (invariant) orthogonal complement of v and $\rho_1 \colon G \to O(1)$ by restriction to v. Since

$$(2.13) \qquad \det(\rho) = \det(\rho_2)\det(\rho_1) = \det(\rho_2)\det(v),$$

the planar symmetry group defined by τ, ρ_2, σ by (2.10) is of type R if and only if $\det(\rho_2) = \det(\tau)$, and hence if an only if $\det(\rho)\det(v) = \det(\tau)$ as claimed. *q.e.d.*

The previous lemma yields the following natural definition.

(2.14) Definition. A space symmetry group G is said *of type R* if it has at least one rotation axis (that is, if it is the extension of a planar group of type R.

(2.15) *Let $\underline{\omega} \in E \cong \mathbb{R}^3$ be a rotation axis for a symmetry group G. Then the Lagrangian action functional $\mathcal{A}_{\underline{\omega}}$ (defined in (2.4)) in a frame rotating around $\underline{\omega}$ with angular speed $\omega = |\underline{\omega}|$ is G-invariant.*

Proof. Let $g \in G$ and $x(t) \in \Lambda^G$. Since for every $i \in \{1, 2, 3\}$

$$x_{gi}(\tau(g)t) = \rho(g)x_i(t),$$

the derivative fulfills the equality

$$\det(\tau(g))\dot{x}_{gi}(\tau(g)t) = \rho(g)\dot{x}_i(t).$$

Thus for every $g \in G$ and $t \in \mathbb{T}$,

$$\dot{x}_{gi}(\tau(g)t) + \Omega x_{gi}(\tau(g)(t)) = \det(\tau(g))\rho(g)\dot{x}_i + \Omega\rho(g)x_i(\tau(t)),$$

6

and hence
$$|\dot{x}_{gi}(\tau(g)t) + \Omega x_{gi}(\tau(g)(t))|^2 = |\dot{x}_i + \det(\tau(g))\rho(g^{-1})\Omega\rho(g)x_i(\tau(t))|^2.$$

One can deduce that the action functional $\mathcal{A}_{\underline{\omega}}$ is G-invariant if (and only if) for every g

(2.16) $$\det(\tau(g))\rho(g^{-1})\Omega\rho(g) = \Omega.$$

If $\Omega \neq 0$, equation (2.16) holds if and only if $\det(\rho_2) = \det(\tau)$, where as above ρ_2 denotes the restriction of ρ to the plane orthogonal to $\underline{\omega}$. But by (2.13) this is equivalent to the identity $\det(\rho)\det(\underline{\omega}) = \det(\tau)$, that is, $\underline{\omega}$ is a rotation axis as in definition (2.11). $\hspace{2cm}$ q.e.d.

(2.17) Definition. A symmetry group G is said *fully uncoercive* if for every possible rotation vector $\underline{\omega}$, the action functional $\mathcal{A}_{\underline{\omega}}^G$ is not coercive.

The following proposition is an easy consequence of the definition and (4.1) of [12].

(2.18) *Let G be a symmetry group.*

   (i) *If there are no rotation axes and $\mathcal{X}^G \neq 0$, (or, equivalently, $\mathcal{A}^G$ is not coercive), then G is fully uncoercive.*

   (ii) *If every rotating axis is uncoercive as a one-dimensional G-module and the action on the index set is not transitive, then G is fully uncoercive.*

(2.19) Definition. If G and G' are two groups conjugated in $O(\mathbb{T}) \times O(3) \times \Sigma_3$, we will write $G \cong G'$. If there exists a change of rotating frame for which a group G can be written as G', which is conjugate to a third group G'', we will write $G \sim G''$. It is easy to see that $\cong$ and $\sim$ are equivalence relations, and that $G \cong G' \implies |G| = |G'|$, while the same does not hold for $\sim$ (see [1], section 3 for further details on changing the coordinates in a rotating frame).

(2.20) Proposition. *Let G be a symmetry group such that a Lagrange rotating solution $x(t) = \{x_j(t)\} = \{e^{ikt}\zeta_3^j\}$ is G-equivariant and $|k+\omega|$ is minimal (as k varies in $\mathbb{Z}$) and not zero. Then $x(t)$ is the absolute minimum of the action functional.*

Proof. This is, for the three-dimensional plane, proposition (4.1) of [1] (see also [8]). Actually, for the three body problem the proof is straightforward: assume that $\sum_i m_i = 1$; since the center of mass is in zero $\sum_i m_i x_i = 0$, and hence $\sum_i m_i \dot{x}_i = 0$, the kinetic energy can be written in terms of the differences

(2.21) $$\frac{1}{2}\sum_i m_i|\dot{x}_i + \Omega x_i|^2 = \frac{1}{2}\sum_{i<j} m_i m_j|\dot{x}_i - \dot{x}_j + \Omega(x_i - x_j)|^2.$$

Thence the action functional is written as the sum of three terms of the type

$$\frac{1}{2}m_i m_j|\dot{x}_i - \dot{x}_j + \Omega x_i - \Omega x_j|^2 + m_i m_j|x_i - x_j|^{-\alpha},$$

which is a Kepler (one-center) problem for the variable $y = x_i - x_j$ in the rotating frame. Since a rotating solution $x(t)$ with $|k + \omega|$ minimal exists by assumption, it yields three (identical, up to a time-shift) rotating solutions in y, with $|k + \omega|$ minimal. It is easy to conclude the proof and to show that every trajectory has an action which is at least three times the action of a minimal one-center y. $\hspace{1cm}$ q.e.d.

7

3 Three-dimensional extensions of planar symmetry groups

In this section we will take planar groups, listed in table 2, and define some extensions acting on the three-dimensional space. Of all the resulting groups, we will take into account only the extensions with trivial core, not bound to collisions and not fully uncoercive. The outcome is the list of table 1. We proceed as follows. Consider one of the planar groups in table 2. It can be extended to a group acting on the three-dimensional space simply by adding a one-dimensional real representation. Now, since the groups have trivial core as for table 2 we can assume that the symmetry group is generated by two elements r and h with the following properties: $\tau(r)$ is a time-shift in $\mathbb{T}$ of minimal angle and $\tau(h)$ is a time-reflection (which exists only if the group is not of cyclic type). Up to conjugacy or change of orientation in $\mathbb{T}$ the choice of r and h yields uniquely back the symmetry group G.

Consider first the case of cyclic type, and let r denote the cyclic generator above (in the notation of table 2, groups of cyclic type are C_1, C_2, C_3 and C_6). Now, $\rho_2(r) \in O(2)$ can be extended in two ways to a matrix in $O(3)$: adding either a trivial one-dimensional representation or a non-trivial one. Thus, for each cyclic group C_i listed above there exist two corresponding groups, denoted by C_i^+ and C_i^-, which are generated by the element $(\tau(r), \rho_2(r), \epsilon, \sigma(r))$ in $O(\mathbb{T}) \times O(2) \times O(1) \times \Sigma_3$, for $\epsilon = \rho_1(r) \in \{\pm 1\}$. The other cases can be dealt with in an analogous way: the choices are 2^2: a sign for $\rho_1(r)$ and a sign for $\rho_1(h)$. So, if G is a symmetry group not of cyclic type, its three-dimensional extensions groups will be denoted by the symbol $G^{\epsilon_1, \epsilon_2}$, where ϵ_1 is the sign of $\rho_1(r)$ and ϵ_2 the sign of $\rho_1(h)$. By (2.12) the third axis will be a rotation axis if and only if the planar symmetry group is of type R. Furthermore, it is easy to see that if the action of the group on the index set is not transitive, extensions of type $+, +, +$ are fully uncoercive. The list of remaining symmetry groups is therefore: C_1^-, $C_1^{+,-}$, $C_1^{-,+}$, $C_1^{-,-}$, $L_2^{+,-}$, $L_2^{-,+}$, $L_2^{-,-}$ C_2^- $H_2^{+,-}$, $H_2^{-,+}$, $H_2^{-,-}$ $H_4^{+,-}$, $H_4^{-,+}$, $H_4^{-,-}$ C_3^-, C_3^- $L_6^{+,+}$, $L_6^{+,-}$, $L_6^{-,+}$, $L_6^{-,-}$, C_6^+, C_6^- $D_6^{+,+}$, $D_6^{+,-}$, $D_6^{-,+}$, $D_6^{-,-}$, $D_{12}^{+,+}$, $D_{12}^{+,-}$, $D_{12}^{-,+}$ and $D_{12}^{-,-}$. Now, some of them are the the same after a change in coordinates: $C_1^{-,-} \cong C_1^{-,+}$, $L_2^{-,+} \cong L_2^{-,-}$, $H_2^{-,+} \cong H_2^{-,-}$, $H_4^{-,+} \cong H_4^{-,-}$, $C_6^+ \cong C_3^-$, $C_6^- \sim C_3$, $D_6^{+,+} \cong L_6^{+,-}$, $D_6^{-,+} \cong D_6^{-,-} \cong D_{12}^{-,-}$, $L_6^{-,+} \cong L_6^{-,-} \cong D_{12}^{+,+}$ and $D_{12}^{-,-} \sim L_6^{+,-}$. Furthermore, $C_1^{+,-}$, $C_1^{-,+}$ and C_2^- are clearly fully uncoercive. Hence the following lemma holds.

(3.1) *Of all the three-dimensional extensions of planar symmetry groups, those with trivial core, not bound to collisions and not fully uncoercive are listed in table 1.*

(3.2) Remark. The order of the space group now does not necessarily coincide with the order of the planar group: for example, the order of C_3^- is 6 and not 3.

The following lemma will be used as a key-step for the classification below.

(3.3) *Let G a symmetry group for the three-body problem with trivial core. Then, up to a change of rotating frame, ρ is the sum of one-dimensional real representations.*

Proof. Since $\ker \tau = 1$, G is isomorphic to a subgroup of a finite dihedral group, and hence its orthogonal irreducible representations have dimension at most 2. So, ρ can

be written as $\rho_2 \times \rho_1$ where $\rho_2 \colon G \to O(2)$ and $\rho_1 \colon G \to O(1)$. Now, by (5.1) of [1] up to a change in rotating frame one can assume that $\rho_2(g)^2 = 1$ for every $g \in G$, so that ρ_2 is reducible as a sum of two one-dimensional G-representations. $\qquad$ q.e.d.

4 Groups without rotation axes

As we have seen in (3.1), the list of candidates for space symmetry groups is given in table 1. Now we consider the 10 groups yielded by extending the three planar groups not of type R.

First, we prove a three-dimensional analogue of proposition (3.9) of [1]. Given a path $x(t) \in \Lambda$, its angular momentum J is the function of $t \in \mathbb{T}$ given by

$$(4.1) \qquad\qquad J(t) = \sum_{i \in \mathbf{n}} m_i x_i \times \dot{x}_i$$

where $\times$ is the vector product in $E \cong \mathbb{R}^3$. If x is a (generalized) solution, then the angular momentum is constant.

(4.2) *For every equivariant $x(t) \in \Lambda^G$ and every $t \in \mathbb{T}$*

$$J(gt) = \det(\rho(g)) \det(\tau(g)) \rho(g) J(t).$$

Proof. It follows from the chain of equalities

$$
\begin{aligned}
J(\tau(g)t) \;&=\; \sum_{i=1}^{3} m_i x_i(\tau(g)t) \times \dot{x}_i(\tau(g)t) \\
&=\; \sum_{i=1}^{3} m_i \left[(\rho(g) x_{g^{-1}i}(t)) \times (\det(\tau(g)) \rho(g) \dot{x}_{g^{-1}i}(t)) \right] \\
&=\; \det(\tau(g)) \sum_{i=1}^{3} m_{g^{-1}i} \left[(\rho(g) x_{g^{-1}i}(t)) \times (\rho(g) \dot{x}_{g^{-1}i}(t)) \right] \\
&=\; \det(\tau(g)) \sum_{i=1}^{3} m_{g^{-1}i} \det(\rho(g)) \rho(g) \left[x_{g^{-1}i}(t) \times \dot{x}_{g^{-1}i}(t) \right] \\
&=\; \det(\tau(g)) \det(\rho(g)) \rho(g) J(t).
\end{aligned}
$$

$\qquad$ q.e.d.

(4.3) *Let $x \in \Lambda^G$ a G-equivariant periodic orbit with angular momentum J. Then J belongs to the subspace $E^* \subset E$ fixed by the G-representation $\det(\rho) \det(\tau)\rho$.*

Proof. By (4.2), for every $g \in G$ $J = \det(\tau(g)) \det(\rho(g)) \rho(g) J$, and hence $J \in E^*$. $\qquad$ q.e.d.

(4.4) *Let G be an extension of a planar symmetry group not of type R. Let $V \subset E$ denote the invariant plane, Then $E^* \subset V$.*

Proof. Let V^* denote the orthogonal complement of the invariant plane. Since $\det(\rho) = \det(\rho_2)\det(\rho_1)$ with $\det(\rho_2)\det(\tau) \neq 1$, the projection of E^* on V^* is fixed by the action of G under the non-trivial homomorphism $\det(\tau)\det(\rho)\det(\rho_1) = \det(\tau)\det(\rho_2)$. Hence $E^* \subset V$. $\hfill q.e.d.$

By (4.4), one needs to consider the vectors in the plane $V \subset E$ fixed by

$$\det(\rho_2(r))\epsilon_1\rho_2(r) \quad \text{and} \quad -\det(\rho_2(h))\epsilon_2\rho_2(h)$$

(the latter only if the action type is not cyclic), where ϵ_1 and ϵ_2 are as above the elements $\rho_1(r)$ and $\rho_1(h)$.

For C_6^+, for example, E^* is the subspace fixed by $-\rho_2(r)$, which is a reflection along a line. Hence C_6^+, even if extension of a planar symmetry group not of type R, might have minimizers with non-zero angular momentum. In fact, it is not difficult to see that up to a change in coordinates $C_6^+ = C_3^-$. On the other hand, for C_6^- it happens that E^* is the subspace fixed by $\rho_2(r)$, which is again a line. Again, as above, C_6^- can be written as C_3^- with a suitable choice of ω for C_3^- (which is of type R).

Now we can consider the extensions of D_6 and D_{12}. For D_6, we have that $\det(\rho_2(r))\epsilon_1\rho_2(r)$ and $-\det(\rho_2(h))\epsilon_2\rho_2(h)$ are respectively equal to ϵ_1 and ϵ_2 (seen as a 2×2 matrices), and hence equivariant minimizers of $D_6^{+,-}$, $D_6^{-,+}$ and $D_6^{-,-}$ have zero angular momentum. On the other hand it is easy to see that after a change of coordinates $D_6^{+,+} = L_6^{+,-}$. For D_{12}, $\det(\rho_2(r))\epsilon_1\rho_2(r)$ and $-\det(\rho_2(h))\epsilon_2\rho_2(h)$ are respectively equal to $-\epsilon_1\rho_2(r)$ (which is a reflection along a line) and ϵ_2 (seen as a matrix). Thus, if $\epsilon_2 = -1$, orbits have zero angular momentum. Otherwise, for $\epsilon_2 = 1$, it is not necessary. Furthermore, it is true that $D_{12}^{+,+} = L_6^{-,-}$ and $D_{12}^{-,+} = L_6^{+,-}$ for a suitable $\underline{\omega}$.

The following definition is the natural extension of the corresponding property for planar groups.

(4.5) Definition. A symmetry group G is said *of type R* if there is a rotation axis for G (and the restriction of the action of G on the invariant plane orthogonal to the axis is a planar symmetry group of type R). A symmetry group G is said to be *not of type R* if there are not rotation axes for G in E.

The following lemma follows immediately from the previous arguments.

(4.6) *If G does not have rotation axes, i.e. it is not of type R, than all G-equivariant trajectories have zero angular momentum and hence they are planar.*

(4.7) *Let G be any space symmetry group not of type R. Then every G-equivariant non-collinear orbit is contained in a (unique) G-invariant plane.*

Proof. By (4.6), the angular momentum is zero and hence the orbit is planar. It is only left to show that the plane containing the orbit is G-invariant. But since for every $g \in G$ and every $t \in \mathbb{T}$

$$\pm g\left[(x_1(t) - x_2(t)) \times (x_1(t) - x_3(t))\right] = (gx_1(t) - gx_2(t)) \times (gx_1(t) - gx_3(t))$$
$$= (x_{g1}(gt) - x_{g2}(gt)) \times (x_{g1}(gt) - x_{g3}(gt)),$$

it follows that the plane containing the configuration $x_1(t)$, $x_2(t)$ and $x_3(t)$ is G-invariant. $\hfill q.e.d.$

5 The vertical variation

Let (z, w) be a system of coordinates for the Euclidean space $E \cong \mathbb{R}^3 \cong \mathbb{C} \oplus \mathbb{R}$, with $z \in \mathbb{C}$ and $w \in \mathbb{R}$. For a planar central configuration $\bar{x}$ consider the planar rotating periodic path $x(t) = e^{ikt}\bar{x}$, with $k \in \mathbb{Z}$. In space the orbit can be written for $i = 1, 2, 3$ as $(x_i(t), w_i(t)) \in \mathbb{R}^2 \times \mathbb{R}$ with $w_i = 0$. Now consider three periodic H^1-functions $\varphi_i \colon \mathbb{T} \to \mathbb{R}$. There is a corresponding path in Λ, which will be denoted with $(x(t), \varepsilon\varphi(t))$, obtained by adding the vertical variation $\varepsilon\varphi$ to the rotation configuration $\bar{x}$.

(5.1) *Let $\mathcal{A}(\varepsilon)$ denote the action of the path $(x(t), \varepsilon\varphi(t))$ in $[0, 2\pi]$; then the second derivative of $\mathcal{A}(\varepsilon)$ evaluated in $\varepsilon = 0$ is*

$$\left.\frac{d^2\mathcal{A}}{d\varepsilon^2}\right|_{\varepsilon=0} = \int_0^{2\pi} \left[\sum_{i\in\mathbf{n}} m_i\dot{\varphi}_i^2 - \alpha \sum_{i<j} \frac{m_i m_j}{|x_i - x_j|^{\alpha+2}}(\varphi_i - \varphi_j)^2\right] dt$$

Proof. The second derivative of the kinetic part is

$$\frac{d^2}{d\varepsilon^2} \sum_{i\in\mathbf{n}} \frac{1}{2}m_i(|\dot{x}_i + \Omega x_i|^2 + \varepsilon^2\dot{\varphi}_i^2) = \sum_{i\in\mathbf{n}} m_i\dot{\varphi}_i^2.$$

Now, it is easy to see that

$$\left.\frac{d^2}{d\varepsilon^2}\right|_{\varepsilon=0} \left[(a + \varepsilon^2 b)^c\right] = 2a^{c-1}bc;$$

moreover, the terms in the potential part contain expressions of such type

$$m_i m_i \left[(x_i - x_j)^2 + \varepsilon^2(\varphi_i - \varphi_j)^2\right]^{-\alpha/2},$$

with $a = (x_i - x_j)^2$, $b = (\varphi_i - \varphi_j)^2$ and $c = -\alpha/2$. Hence

$$\left.\frac{d^2}{d\varepsilon^2}\right|_{\varepsilon=0} \sum_{i<j} m_i m_j |(x_i, \varphi_i) - (x_j, \varphi_j)|^{-\alpha} = \sum_{i<j} m_i m_j 2\left[(x_i - x_j)^2\right]^{-\alpha/2-1}(\varphi_i - \varphi_j)^2(-\frac{\alpha}{2})$$

$$= -\alpha \sum_{i<j} \frac{m_i m_j}{|x_i - x_j|^{\alpha+2}}(\varphi_i - \varphi_j)^2.$$

Thus, the claim. *q.e.d.*

(5.2) *Consider the path $x(t) = e^{ikt}\bar{x}$ as above. For a unit vector $\mathbf{e} \in \mathbb{C} \subset E$ define $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ by the scalar product $\varphi_i(t) = x_i(t/k) \cdot \mathbf{e}$ for $i = 1, 2, 3$. Then the second variation of (5.1) is*

$$\left.\frac{d^2\mathcal{A}}{d\varepsilon^2}\right|_{\varepsilon=0} = \pi\left(I(\bar{x}) - \alpha U(\bar{x})\right),$$

where $I(\bar{x}) = \sum_i m_i\bar{x}^2$ is the momentum of inertia of $\bar{x}$ and $U(\bar{x})$ the value of the potential function.

Proof. Define $\beta = 1/k$; then $\varphi_i(t) = x_i(\beta t) \cdot \mathbf{e} = (e^{it}\bar{x}_i) \cdot \mathbf{e}$ and therefore

$$\int_0^{2\pi} m_i\dot{\varphi}_i^2 \, dt = \int_0^{2\pi} m_i\bar{x}^2 \sin^2(t + \delta_i) \, dt = \pi m_i\bar{x}^2$$

for some suitable δ_i, which implies that

$$\int_0^{2\pi} \sum_{i=1}^3 m_i\dot{\varphi}_i^2 = \pi \sum_{i=1}^3 m_i\bar{x}^2 = \pi I(\bar{x}).$$

As for the second part of the expression in (5.1), since the norms $|x_i - x_j|$ are constant one obtains

$$
\begin{aligned}
\int_0^{2\pi} \frac{m_im_j}{|x_i(t) - x_j(t)|^{\alpha+2}}(\varphi_i - \varphi_j)^2 \, dt &= \int_0^{2\pi} \frac{m_im_j}{|x_i(t) - x_j(t)|^{\alpha}} \left(\frac{\varphi_i - \varphi_j}{|x_i(t) - x_j(t)|} \right)^2 dt \\
&= \frac{m_im_j}{|\bar{x}_i - \bar{x}_j|^{\alpha}} \int_0^{2\pi} \left(\frac{x_i(\beta t) - x_j(\beta t)}{|x_i(\beta t) - x_j(\beta t)|} \cdot \mathbf{e} \right)^2 dt \\
&= \frac{m_im_j}{|\bar{x}_i - \bar{x}_j|^{\alpha}} \int_0^{2\pi} \cos^2(t + \delta_{ij}) \, dt \\
&= \pi \frac{m_im_j}{|\bar{x}_i - \bar{x}_j|^{\alpha}}.
\end{aligned}
$$

with a suitable choice of the shift constant δ_{ij}. Thus, summing up one obtains

$$\int_0^{2\pi} \sum_{i<j} \frac{m_im_j}{|x_i - x_j|^{\alpha+2}} (\varphi_i - \varphi_j)^2 \, dt = \pi \sum_{i<j} \frac{m_im_j}{|\bar{x}_i - \bar{x}_j|^{\alpha}} = \pi U(\bar{x})$$

The conclusion follows. *q.e.d.*

Until now we did not assume anything else on $x(t)$ other than being rotating k times during the interval $[0, 2\pi]$. Now we assume that it is a minimizer in a suitable *linear* class of paths (such as Λ^G for a suitable G acting on the plane or the space). Then the following equation holds

(5.3) *If $x(t) = e^{ikt}\bar{x}$ is a minimizer of $\mathcal{A}_\omega$, then (Kepler's law)*

$$(k + \omega)^2 I(\bar{x}) = \alpha U(\bar{x}).$$

Proof. It is easy to see that the action is

$$\frac{1}{2\pi}\mathcal{A}_\omega = \frac{1}{2}(k + \omega)^2 I(\bar{x}) + U(\bar{x}).$$

By deriving the expression in $R = \sqrt{I}$ one obtains that the minimum as $R > 0$ varies is achieved when $(k + \omega)^2 R^2 = \alpha U(\bar{x})$. Otherwise, one could also use homogeneity and directly Newton's equations. *q.e.d.*

(5.4) Proposition. *Assume that for a symmetry group G every rotating G-equivariant central configuration $x(t) = e^{ikt}\bar{x}$ is such that $(k + \omega)^2 > 1$. Then rotating central configurations cannot be minimizers of $\mathcal{A}^G$.*

Proof. By (5.2) the second variation is $\dfrac{d^2\mathcal{A}}{d\varepsilon^2}\Big|_{\varepsilon=0} = \pi(I(\bar{x}) - \alpha U(\bar{x}))$. But by (5.3)

$$\alpha U(\bar{x}) = (k+\omega)^2 I(\bar{x}),$$

so that

$$\frac{d^2\mathcal{A}}{d\varepsilon^2}\Big|_{\varepsilon=0} = \pi I(\bar{x})(1 - (k+\omega)^2) < 0,$$

which shows that $x(t)$ cannot be a minimizer. *q.e.d.*

Now we consider a different vertical variation, which can be readily used for a vertical isosceles triangle. Consider now the variation φ (of (5.1)) defined as follows: $\varphi = v \sin t$, where $v \in \mathbb{R}^n$ is a one-dimensional configuration with $\sum_{i\in\mathbf{n}} m_i v_i = 0$. Without loss of generality we assume that $\sum_{i\in\mathbf{n}} m_i = 1$.

(5.5) *Let G be a symmetry group and $x(t) = e^{ikt}\bar{x}$ a planar G-equivariant rotating central configuration. If (x, φ) is G-equivariant and*

$$(5.6) \qquad \sum_{i<j} m_i m_j (v_i - v_j)^2 \left(1 - \alpha|\bar{x}_i - \bar{x}_j|^{\alpha+2}\right) < 0,$$

then $x(t)$ is not a minimizer.

Proof. Since $\sum_{i\in\mathbf{n}} m_i \varphi_i = 0$ and $\sum_{i\in\mathbf{n}} m_i = 1$ by assumption, one can write as in (2.21) the kinetic energy in terms of differences, and therefore equation (5.1) can be read as

$$\frac{d^2\mathcal{A}}{d\varepsilon^2}\Big|_{\varepsilon=0} = \pi\left[\sum_{i\in\mathbf{n}} m_i m_j (v_i - v_j)^2 - \alpha \sum_{i<j} \frac{m_i m_j}{|x_i - x_j|^{\alpha+2}} (v_i - v_j)^2\right],$$

since $\int_0^{2\pi} \sin^2 t = \int_0^{2\pi} \cos^2 t = \pi$, and this implies the claim. *q.e.d.*

(5.7) Proposition. *In the hypotheses of (5.5), consider the following case: $n = 3$, $m_1 = m_2$, $x_1 = -x_2$ (and hence $x_3 = 0$) and $v_1 = v_2$. Then $\dfrac{d^2\mathcal{A}}{d\varepsilon^2}\Big|_{\varepsilon=0} < 0$ if and only if*

$$(5.8) \qquad (k+\omega)^2 > \frac{m_1}{2^{\alpha+1}} + 1 - 2m_1.$$

Proof. If $x(t) = e^{ikt}\bar{x}$ is a minimum, then $\bar{x} = (R, -R, 0)$, with

$$(5.9) \qquad R^{\alpha+2} = \frac{\alpha}{2\beta},$$

where $\beta = \dfrac{(k+\omega)^2}{m_1 2^{-\alpha} + 2m_3}$. Since $v_1 = v_2$, the left term of (5.6) is equal to

$$2m_1 m_3 (v_1 - v_3)^2 \left(1 - \alpha|\bar{x}_1 - \bar{x}_3|^{\alpha+2}\right),$$

and therefore $\dfrac{d^2\mathcal{A}}{d\varepsilon^2}\Big|_{\varepsilon=0} < 0$ if and only if $R^{\alpha+2} < \alpha$. But by (5.9) this is true if and only if

$$(k+\omega)^2 > \frac{m_1}{2^{\alpha+1}} + 1 - 2m_1,$$

as claimed. *q.e.d.*

Table 3: The list of all dihedral symmetries

000: $\begin{bmatrix} + & + \\ + & + \\ + & + \end{bmatrix}$
001: $\begin{bmatrix} + & + \\ + & + \\ + & - \end{bmatrix}$
002: $\begin{bmatrix} + & + \\ + & + \\ - & + \end{bmatrix}$
003: $\begin{bmatrix} + & + \\ + & + \\ - & - \end{bmatrix}$
011: $\begin{bmatrix} + & + \\ + & - \\ + & - \end{bmatrix}$

012: $\begin{bmatrix} + & + \\ + & - \\ - & + \end{bmatrix}$
013: $\begin{bmatrix} + & + \\ + & - \\ - & - \end{bmatrix}$
022: $\begin{bmatrix} + & + \\ - & + \\ - & + \end{bmatrix}$
023: $\begin{bmatrix} + & + \\ - & + \\ - & - \end{bmatrix}$
033: $\begin{bmatrix} + & + \\ - & - \\ - & - \end{bmatrix}$

111: $\begin{bmatrix} + & - \\ + & - \\ + & - \end{bmatrix}$
112: $\begin{bmatrix} + & - \\ + & - \\ - & + \end{bmatrix}$
113: $\begin{bmatrix} + & - \\ + & - \\ - & - \end{bmatrix}$
122: $\begin{bmatrix} + & - \\ - & + \\ - & + \end{bmatrix}$
123: $\begin{bmatrix} + & - \\ - & + \\ - & - \end{bmatrix}$

133: $\begin{bmatrix} + & - \\ - & - \\ - & - \end{bmatrix}$
222: $\begin{bmatrix} - & + \\ - & + \\ - & + \end{bmatrix}$
223: $\begin{bmatrix} - & + \\ - & + \\ - & - \end{bmatrix}$
233: $\begin{bmatrix} - & + \\ - & - \\ - & - \end{bmatrix}$
333: $\begin{bmatrix} - & - \\ - & - \\ - & - \end{bmatrix}$

(5.10) *Remark.* The right hand side of (5.8) is a linear function of m_1, which is defined for $m_1 \in (0, 1/2)$ and goes from a limit value of 1 (for $m_1 = 0$) to a limit value of $2^{-(2+\alpha)}$ (for $m_1 = 1/2$). Hence it is always possible to find mass distributions (m_1, m_2, m_3) for which minimizers are not rotating Euler solutions, provided that for the minimal k one has $(k + \omega)^2 > 2^{-(2+\alpha)}$. For example, if $\alpha = 1$ then one finds that there is a non-trivial interval of values of ω for which minimizers are non-trivial for all $m \in (\frac{3}{7}, \frac{1}{2})$ (in the case that k can be any integer) while the same happens and for all $m \in (0, \frac{1}{2})$ if the symmetry group implies a constraint on k such that $k = 0$ mod 2. For equal masses one has $m = \frac{1}{3} \notin (\frac{3}{7}, \frac{1}{2})$ and so one has to assume $k = 0$ mod 2, inequality (5.8) becomes

$$(k + \omega)^2 > \frac{5}{12},$$

and thus non-planar orbits exist for $\omega \in (\sqrt{\frac{5}{12}}, 2 - \sqrt{\frac{5}{12}})$. We will apply this simple argument below in (7.7) to prove the existence of non-planar (quasi-)periodic orbits for when two masses are approximately equal.

6   Space symmetries

In this section we will describe space symmetries and prove theorem A.

Consider the case of groups with trivial core. Let $r \subset G$ denote the $\mathbb{T}$-cyclic generator, and, if it exists, let $h \in G$ denote one of the time reflections. Consider $r_\Sigma = \sigma(r)$, $h_\Sigma = \sigma(h)$, $r_V = \rho(r)$ and $h_V = \rho(h)$. By (3.3), the G-representation ρ is the sum of one-dimensional components, hence r_V and h_V can be written as

14

Table 4: Symmetry groups of cyclic type

	()	(1, 2)	(1, 2, 3)
$[+++]$	C_1^+ (fully uncoercive)	C_2^+ (fully uncoercive)	C_3^+
$[++-]$	C_1^-	C_2^- (fully uncoercive)	C_3^-
$[+--]$	L_2^- (fully uncoercive)	H_2^- (fully uncoercive)	$C_6^- \sim C_3^+$
$[---]$	$\sim C_1^-$	$\sim C_2^-$	$\sim C_3^-$

matrices with ± 1 diagonal elements. Thus a choice of the generators r and h yields
a 3×2 matrix $\begin{bmatrix} r_V{}^1 & h_V{}^1 \\ r_V{}^2 & h_V{}^2 \\ r_V{}^3 & h_V{}^3 \end{bmatrix}$ where the entries $r_V{}^i$ and $h_V{}^i$ are the diagonal entries
of the matrices r_V and h_V respectively. Conversely, if such a matrix is given, the
elements r and h can be obtained by the permutations r_Σ and h_Σ in Σ_3 (analogously
for cyclic and brake action types). The number of such matrices is the number of
unordered 3-tuples of elements chosen in the set $\{[++], [+-], [-+], [--]\}$, which are
$\binom{4+3-1}{3} = 20$. Under the identification $0 = [++], 1 = [+-], 2 = [-+]$ and $3 = [--]$,
it is possible to represent such matrices with numbers of 3 digits as in table 3.

If the group G is of cyclic type, then there are 12 possible cases for r_V and r_Σ
(4 for r_V times 3 for r_Σ): $r_\Sigma \in \{(), (1,2), (1,2,3)\}$ and $r_V \in \{[+++], [++-], [+--], [---]\}$. Is not difficult to see that the resulting groups are three-dimensional
extensions listed in table 4 (where, as we defined in (2.19) from now on the symbol
'$\sim$' means that the symmetry group is equivalent to the group in question after a
change in rotating coordinates). Given a symmetry group G, consider the elements
r_V, r_Σ, h_V and h_Σ defined above. The matrix $\begin{bmatrix} r_V{}^1 & h_V{}^1 \\ r_V{}^2 & h_V{}^2 \\ r_V{}^3 & h_V{}^3 \end{bmatrix}$ associated to r_V and
h_V is one of the 20 matrices of table 3. Furthermore, it is easy to prove that up to
permutations the pair $[r_\Sigma, h_\Sigma]$ can be chosen from the set

$$\{[(1,2,3), (1,2)], [(), (1,2)], [(1,2), ()], [(1,2), (1,2)], [(), ()]\}.$$

Consider first the case $[r_\Sigma, h_\Sigma] = [(1,2,3), (1,2)]$. Since the matrices

(6.1)
$$\begin{bmatrix} r_V{}^1 & h_V{}^1 \\ r_V{}^2 & h_V{}^2 \\ r_V{}^3 & h_V{}^3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} r_V{}^1 & h_V{}^1 r_V{}^1 \\ r_V{}^2 & h_V{}^2 r_V{}^2 \\ r_V{}^3 & h_V{}^3 r_V{}^3 \end{bmatrix}$$

yield the same symmetry group up to change of coordinates as the matrix it is possible
to assume that $[r_V, h_V]$ belongs to one of the 13 items: 000, 001, 002 ($\cong$ 003), 011,
012 ($\cong$ 013), 022 ($\cong$ 033), 023, 111, 112 ($\cong$ 113), 122 ($\cong$ 133), 123, 222 ($\cong$ 333), 223
($\cong$ 233).

Now, since we are excluding the case of groups which are bound to collisions, we
have to rule out the cases in which h_V or the product $r_V h_V$ is trivial, which means
we have to exclude the four cases 000, 002 $\cong$ 003, 022 $\cong$ 033, 222 $\cong$ 333; thus we are
left with a list of 9 matrices.

We can now take the rotation axes into account. Let e_1, e_2 and e_3 denote the
canonical elements of the base of the vector space E. In table 5 it is found the

15

001	011	111	012	112	023	122	123	223
e_1, e_2	e_2, e_3		e_3		e_1		e_1	e_2, e_3

Table 5: Rotation axes

list the elements of $\{e_1, e_2, e_3\}$ which are rotation axes for the corresponding group. Using the rotating frame change of coordinates, it is therefore possible to show that $023 \sim 001$, $123 \sim 011$ and $223 \sim 012$, and hence that we are left with a choice among the 6 items $001 = L_6^{+,-}$, $011 = D_6^{+,+} \cong L_6^{+,-}$, $111 = D_6^{+,-}$, $012 = D_{12}^{+,+} \cong L_6^{-,+}$, $112 = D_6^{-,+}$ and $122 = D_{12}^{-,+}$. Consider now the case $[r_\Sigma, h_\Sigma] = [(), (1, 2)]$. We shall proceed in a similar way, analyzing case by case until we are left with a small number of significative choices. As above, 000, 002, 022 and 222 yield a group which is bound to collisions, and up to a change of coordinates we can choose among the same following 9 items: 001, 011, 012 ($\cong$ 013), 023, 111, 112 ($\cong$ 113), 122 ($\cong$ 133), 123, 223 ($\cong$ 233).

The rotation axes are the same as those of table 5 above, and again in a suitable rotating frame $023 \sim 001$, $123 \sim 011$ and $223 \sim 012$, so that we can choose just among the 6 items 001, 011, 111, 012, 112 and 122. First an easy computation shows that 111, 112 and 122 yield symmetry groups without rotation axes and at the same time not coercive (thus fully uncoercive by (2.18)). Furthermore, the choice of 001, 011 or 012 yields groups G conjugate to the three-dimensional extensions $H_2^{+,+}$, $H_2^{+,-}$ and $H_2^{-,+}$ of the planar Isosceles symmetry group H_2). As a third possibility, now consider the cases $[r_\Sigma, h_\Sigma] = [(1, 2), (1, 2)]$ and $[r_\Sigma, h_\Sigma] = [(1, 2), ()]$. We can consider just the case $[r_\Sigma, h_\Sigma] = [(1, 2), (1, 2)]$, up to a change of coordinates (but we will not be able to use the argument of (6.1) to reduce the number of matrices). As above, we start with the list 3 of all possibilities. Since $h_\Sigma = (1, 2)$, if h_V is trivial then the resulting group is bound to collisions, and therefore we cancel the four matrices 000, 002, 022 and 222. From the 16 matrices left the following 8 do not have a rotation axis: 003, 033, 111, 112, 113, 122, 133 and 333. Since the action on $\{1, 2, 3\}$ is not transitive, by (2.18) all those with a row equal to $[++]$ are fully uncoercive (that is, 003 and 033). Furthermore, for the matrices 111, 112 and 122 the resulting symmetry group is bound to collisions (since $r_V h_V$ is the antipodal map, while $r_\Sigma h_\Sigma$ is the trivial permutation). About the three remaining items 113, 133 and 333, they are fully uncoercive simply because they contain a row equal to $[--]$ (which yields a one-dimensional non-coercive symmetry group).

So, we are left with 8 choices, all with rotation axes: 001, 011, 012, 013, 023, 123, 223, 233. After a change in rotating coordinates one can see that $023 \sim 001$, $123 \sim 011$, $223 \sim 012$ and $233 \sim 013$; furthermore, it is easy to see that 001 and 013 yield fully uncoercive symmetry groups. As a consequence, the remaining matrices are 011 and 012. In the notation of table 1, they are respectively the symmetry groups $H_4^{+,-}$ and $H_4^{-,+}$. At last, we can consider the case $[r_\Sigma, h_\Sigma] = [(), ()]$ where the resulting group acts trivially on the index set. The matrices 111, 113, 133 and 333 yield bound to collisions groups. By the same argument as (6.1), we do not consider the duplicates 003, 013, 033, 112, 122, 222, 233. Of the remaining 9 matrices, three do not have rotation axes (000, 002 and 022) and yield groups which are not coercive, while by a change in rotating frames the other six can be reduced to the three 001($\sim$ 023), 011($\sim$ 011) and 012($\sim$ 012). The group induced by 001 is fully

uncoercive. So, we can as before reduce the list of 20 matrices to the two cases 011 and 012 which yield respectively the groups $L_2^{+,-}$ and $L_2^{-,+}$ of table 1.

(6.2) Definition. We say that a symmetry group is of *vertical isosceles* type if its core is generated by an element k such that $\rho(k)$ is the rotation of angle π around an axis and $\sigma(k)$ is conjugate to the permutation $(1,2)$.

To conclude the proof it is left to prove the easy fact that if the core is non-trivial, then the group is homographic, provided that it is not of vertical isosceles type.

(6.3) *Let G be a symmetry group which is not bound to collisions and not fully uncoercive. Then either G-equivariant trajectories are always homographic, or the group is of vertical isosceles type (6.2).*

Proof. As in the proof of the (similar) proposition 5.4 for planar groups of [1], $\ker \tau$ is isomorphic to a subgroup of Σ_3 and hence its three-dimensional representation in E is reducible: that is, there is a $\ker \tau$-invariant line $\mathbb{R}$ in E. Now, since $\ker \tau$ is normal in G, either $g\mathbb{R} \subset \mathbb{R}$ for every $g \in G$, or E is the sum of three copies of $\mathbb{R}$, which are permuted by the elements in G (hence $\ker \tau$ has order 2 and acts as the antipodal map -1 on E). After an analysis of a few cases, it is clear that the possible actions of the core are the following: $\ker \tau = \langle (a_3, (1,2)) \rangle$, $\ker \tau = \langle (r_2, (1,2)) \rangle$, $\ker \tau = \langle (r_3, (1,2,3)) \rangle$, $\ker \tau = \langle (r_3, (1,2,3)), (h_2, (1,2)) \rangle$ (which, incidentally, are extensions of planar analogues), where a_3 is the antipodal map $a_3 = -1$, r_2 the rotation of π around a fixed axis, r_3 the rotation of $2\pi/3$ around a fixed axis $\mathbb{R}$, and h_2 the reflection with respect to a plane containing (or with respect to a line orthogonal to $\mathbb{R}$). In the first case $\ker \tau$-invariant configurations are antipodal binaries with a third mass at the origin (this is equivalent to a spatial Kepler problem), which is a homographic group; the second is the well-known case of isosceles triangle (which is not homographic); the third and the fourth yield homographic symmetry groups. Thus the proof. *q.e.d.*

7   Proof of theorem B

In this section we will complete the proof of B, proving one-by-one all its parts.

(7.1) *Let G be a symmetry group of the three-body problem which is not bound to collisions. Then all G-equivariant minimizers are collisionless.*

Proof. Consider first the case of a group with trivial core; let H be one of its maximal $\mathbb{T}$-isotropy groups: H is generated by the non-trivial element $h \in H$ with $\sigma(h) \in \{(), (1,2)\}$. The orthogonal motion $\rho(h)$ is of order at most two, and hence there are at least three invariant orthogonal lines (equivalently, the representation of H is the sum of one-dimensional H-representations). An immediate consequence is that if it is not bound to collisions (that is, if $(\rho(h), \sigma(h)) \neq (1, (1,2))$, the subgroup H has the rotating circle property (10.1) of [12]. Thus, by (10.10) of the same paper, G-equivariant minimizers are collisionless. Now assume that the kernel of τ is not trivial. As in the proof of (6.3) it is possible to assume that $\ker \tau$ acts as one of the four cases listed, where the first case yields a one-center (Kepler) problem in space,

the third and fourth cases yield a one-center planar problem, and the second case is the isosceles triangle, see (6.2). One can readily see that theorem (10.1) of [12] can be applied in all the cases, and hence the thesis. *q.e.d.*

(7.2) *For every ω, minimizers for C_1^-, $H_2^{+,-}$, C_3^+, $L_6^{+,+}$, $L_6^{+,-}$ are Lagrange homographic solutions.*

Proof. It is easy to see that the hypothesis of (2.20) are fulfilled, hence the thesis. *q.e.d.*

(7.3) *Minimizers for $D_6^{+,-}$, $D_6^{-,+}$ and $D_{12}^{-,+}$ are zero-angular momentum planar solutions: the Chenciner-Montgomery figure-eight solution.*

Proof. Since these groups do not have rotation axes, by lemma reflemma:nottypeR, their minimizers are planar orbits with zero angular momentum, contained in a G-invariant plane by (4.7). All planes are G-invariant for $D_6^{+,-}$, and the restricted group is D_6. Hence by (4.15) of [1] the minimizer for $D_6^{+,-}$ is the D_{12}-symmetric Chenciner-Montgomery figure eight [10]. Next, if $G = D_{12}^{-,+}$, a G-orbit (which is collisionless by (7.1), has to be contained in the G-invariant plane where G acts as D_{12}, since otherwise it would result to be bound to collisions. Hence the minimum for $D_{12}^{-,+}$ is again Chenciner-Montgomery figure eight. At last, consider $D_6^{-,+}$, which is a group of order 12. In any one of the infinitely many invariant planes with D_{12}-action as restriction there is a CM-eight minimum, while in the other invariant plane there is a redundant D_6-action (hence if the minimum were to be contained in this plane, it would have implied that the action of the D_6-eight is less than the action of the D_{12}-eight, which is not true by [1]). Thus as above the minimum is the D_{12}-symmetric CM-eight. *q.e.d.*

(7.4) *For $\omega \in (-1, 1) + 6\mathbb{Z}$, minimizers for $L_6^{-,+}$ and its subgroup $C_3^- \subset L_6^{-,+}$ are the non-planar (if $\omega \neq 0$) families of quasi-periodic solutions called respectively P_{12} and P'_{12}, which might be likely to coincide.*

Proof. By (7.1), minima for C_3^- and $L_6^{-,+}$ exist and are collisionless. If they are planar, then in both cases they have to be a Lagrange rotating solution (rotating at angular speed k, with $k = \pm 2 \mod 6$, minimizing the number $(k + \omega)^2$). The proof can be concluded by applying (5.4). Action levels for the resulting minima are depicted in figure 1: in the intervals $(1/6, 1/2)$ and $(1/2, 5/6)$ the minimum is a rotating Lagrange solution, while otherwise it is a non-planar orbit (in the graph the period is rescaled to 12π, so that values of ω need to be multiplied by a factor 6). It is remarkable how the estimate of (5.4) seems to be sharp, since when its hypothesis is not fulfilled one can find numerically that minimizers are in fact planar rotating Lagrange triangles. *q.e.d.*

(7.5) Remark. In $[14]^2$ Marchal actually introduced, under a different notation, the family corresponding to minimizers (as ω varies) under $L_6^{-,+}$-symmetries, naming

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[2]The proof of the existence, claimed in [14] and later in [15], was completed in [7, 5] by action level estimates on colliding paths, since the main result of [15] cannot be applied to the P_{12} family, which has a symmetry group of dihedral type. The action levels graph of [7] is a qualitative picture of figure 1

Figure 1: Action-levels for the $L_6^{-,+}$-symmetric P_{12} minimizers
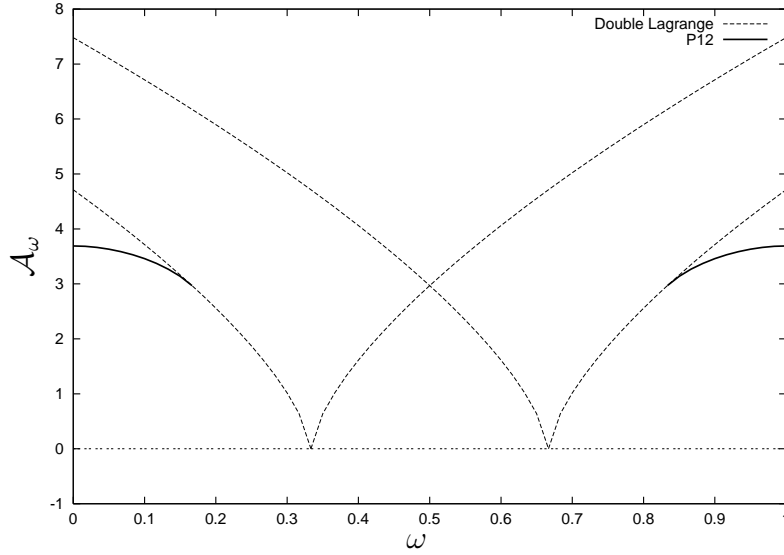
it P_{12} – see also [13]. The fact that there is also a C_3^--equivariant family (as well as the fact that there is a group action of cyclic type yielding a figure-eight orbit) apparently was not known. As for the planar eights, it seems that (numerically) these two families coincide, and that for $\omega = 0$ one finds the CM-eight and the cyclic eight (which should as well likely coincide). Some of these were questions raised in the last section of [14] (see also section 4.(iv) of [5]), questions which probably still need to find an answer. For example, as said at the end of section 4 of [5], one should prove that for $\omega = 0$ the minimum is planar and that minimizers are a continuous family (from figures 1 and 2(a) it seems that action levels depend continuously on ω, as well as the corresponding trajectories). In figure 2(a) the minimizers corresponding to the values $\omega = j/5$, for $j = 0 \ldots 5$ are shown, together with their projections. The curves with $j = 0$ and $j = 5$ have a bigger width. On its side, there is one of the orbits (actually, corresponding to the non-integer value $j = 2.5$) in the inertial frame.

(7.6) Remark. In [9], Chenciner, Féjoz and Montgomery found, under an assumption (at the moment numerically evaluated) of non-degeneracy for the CM-eight, three families of periodic orbits in rotating frames, originated by three different symmetry-breaks of the planar eight. The family there termed Γ_1 is P_{12}.

(7.7) *For every $\omega \notin \mathbb{Z}$ there is a mass distribution such that minimizers of $L_2^{+,-}$, $L_2^{-,+}$, $H_4^{+,-}$ and $H_4^{-,+}$ (where for the first two masses do not need to be equal, while in the second three two masses need to be equal) are not planar. Conversely, for every mass distribution with two equal masses there is an ω such that minimizers symmetric under the groups $L_2^{-,+}$ are not planar.*

Proof. If follows directly from remark (5.10). q.e.d.

(a) Some representatives of the P_{12} family in the rotating frame

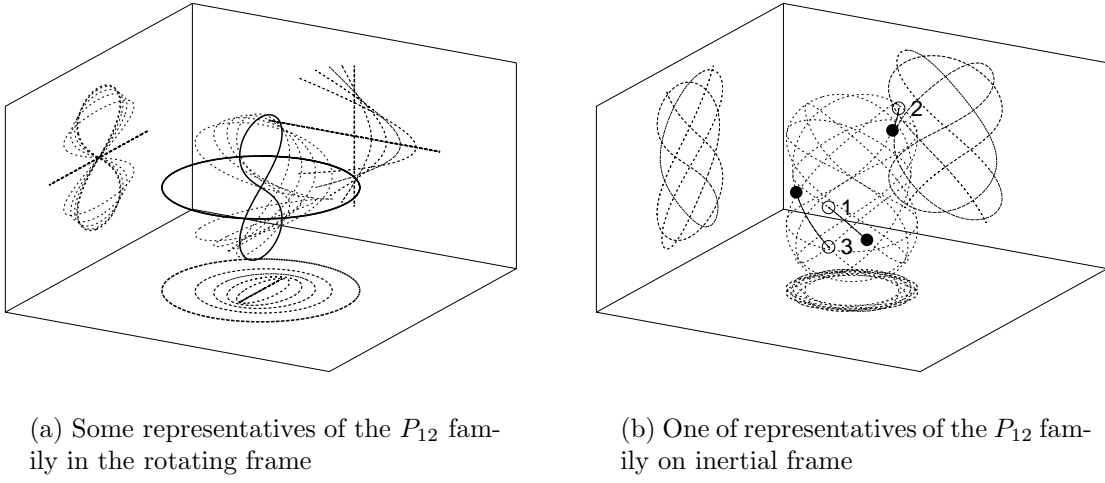(b) One of representatives of the P_{12} family on inertial frame

Figure 2: The P_{12} family

8 Remarks

We did not prove general and complete results about planarity or non-planarity of minimizers for the following symmetry groups: $L_2^{+,-}$, $L_2^{-,+}$, $H_2^{-,+}$, $H_4^{+,-}$ and $H_4^{-,+}$. In fact, we have proved that for some choices of masses and angular speed minima are non-planar, but we could not prove that minima are planar for other choices (as apparently they are: all the Hill-type orbits and Euler solutions exposed in for example [1]). We describe now, among other remarks, some properties of their minimizers with a little bit more details.

(8.1) If the masses are equal, for all ω minimizers under the symmetry group $L_2^{+,-}$ are planar (and they are the Euler and Hill retrograde orbits described in [1] – note that it is easy to prove by (5.7) that for all these symmetry groups there are choices of masses and angular speed ω for which minimizers are not planar). Also, it turns out that there are other local minimizers (planar and non-planar).

(8.2) The symmetry group $L_2^{-,+}$ imposes that possible rotating configurations (which are Euler collinear) have to rotate an even number of loops, i.e. the cyclic part of the symmetry group imposes that a rotating central configuration has to have $k = 0$ mod 2, and hence by (5.10) there is a continuum of choices for ω, for every choice of non-zero masses, such that the minimizer is non-planar. An example is shown in figure 3.

(8.3) If the symmetry group is $H_2^{-,+}$, then again it implies that a rotating central configuration has to rotate a number $k = 0$ mod 2 of times. The possible rotating configuration is a Lagrange triangle and it is not possible to apply (5.4) to show that it is not a minimum since it is always possible to find k with $|k + \omega| \leq 1$. On the other hand the constraint $k = 0$ mod 2 prevents one to apply (2.20) to show that the minimum is in fact a Lagrange solution. Numerical experiments show that this is the case.

(8.4) Now consider the symmetry groups $H_4^{+,-}$ and $H_4^{-,+}$. As in the planar case, one finds non-homographic minimizers for some interval of values of ω (and approximately

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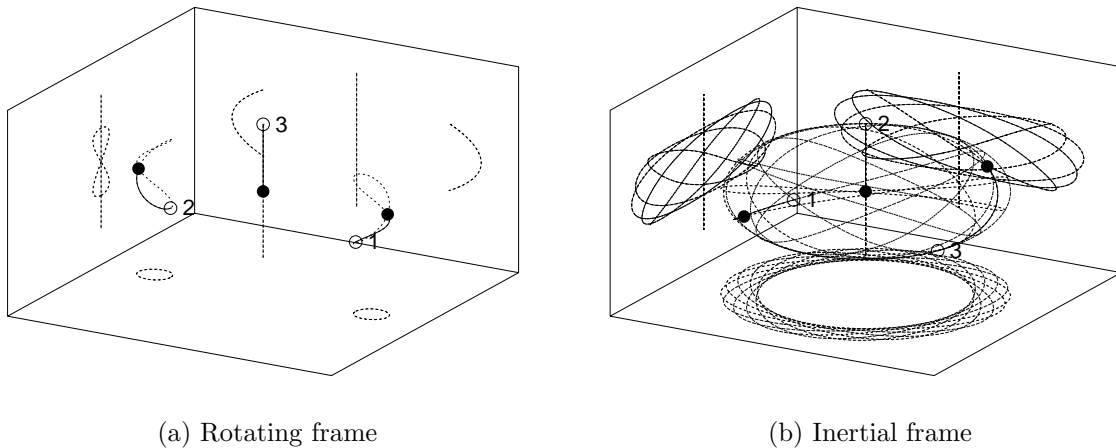
(a) Rotating frame   (b) Inertial frame

Figure 3: A non-planar $L_2^{-,+}$-symmetric orbit with equal masses, $\omega = 0.9$

equal masses – see figure 4 of [1]). Homographic solutions have to be Euler-Moulton rotating collinear configurations, and hence we can literally repeat the arguments applied above to $L_2^{+,-}$ and $L_2^{-,+}$, and obtain the fact that for two equal masses it is always possible to find intervals of angular speed ω for which minimizers are not planar. This time rotating central configurations do not necessarily have $k = 0$ mod 2, since H_4 has already a cyclic part of order 2, and hence one cannot prove with the vertical variation above that there are non-planar orbit for equal masses (for equal masses, after numerical experiments it seems that local minima under $H_4^{+,-}$ are planar, while local minima under the action of $H_4^{-,+}$ can be non-planar for some ω).

(8.5) In the proofs of (5.4) and (5.5) there is clearly no need to assume the bodies to be three. In fact, the same vertical variation yield interesting non-planar orbits for every number of bodies. For example, it is easy to show by a straightforward extension of (5.2) that families corresponding to the P_{12} and P'_{12} ones exist for any odd number n of bodies. In fact, consider the cyclic group C of order $2n$, acting by a cyclic permutation of the n bodies on the index set and by a reflection along a plane p in the space E. Then, C-symmetric loops are choreographies in E consisting in n bodies, and if we choose as rotation axis $\underline{\omega}$ the line orthogonal to the plane p, we obtain family of coercive n-body problems such that for $\omega \in (-1, 1) \mod 2n$ minima are not equilibrium solutions. Since they are collisionless due to [12], they are periodic orbits (non-planar for $\omega \neq 0$). It is likely that they behave like the P_{12}, namely that they connect a eight solution with a (twice rotating in the rotating frame) homographic solution, but this is probably hard to prove (already for $n = 3$ nobody has published a proof that for $\omega = 0$ the P_{12} is a planar eight and that the family is a continuous one).

(8.6) About (5.4), a similar proposition was used by Chenciner in [6] to show that minimizers for the $n \geq 4$ anti-symmetric loops are non-planar, following Moeckel's theorem on central configurations [16]. While the computation is very similar, here we use a different type of vertical variation, which yield solutions in particular also
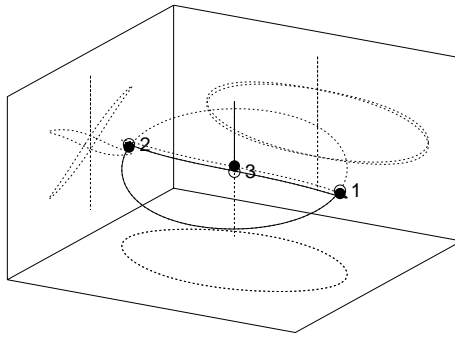
21

Figure 4: The simplest non-planar solution with three equal masses

when the rotating central configuration minimizes the reduced potential $\tilde{U}$, due to the greater number of loops that symmetry constraints impose on rotating central configurations. Furthermore, in section 3 of [6] there is an interesting short remark proving the existence of a non-planar periodic solution for 3 bodies in the vertical isosceles problem under the antisymmetry constraint. Since this constraint coincides with the group C_1^- with angular speed $\omega = 1$ (group which implies $k = 0 \mod 2$ to any C_1^--equivariant equilibrium solution), one can use (5.7) and obtain that for all $\omega \in (\sqrt{\frac{5}{12}}, 2 - \sqrt{\frac{5}{12}})$ (as in remark (5.10) above) and equal masses a C_1^--symmetric vertical isosceles minimizer is not planar. In figure 4 it is shown the solution in the inertial frame – probably this is the simplest non-planar periodic solution of the three-body problem.

(8.7) In this paper we have sometimes rescaled the period to a number different than 2π. It is worth mentioning that, because of the homogeneity of the potential, a minimizer in a frame rotating with angular velocity ω and period $k2\pi$ corresponds to a minimizer with period 2π in a frame rotating with angular velocity $k\omega$. The reason of rescaling the period is that in our numerical experiments we have decided to rescale the period in order to have a fundamental domain of length π. The data for the minimizers used for figures were obtained by a custom optimization program running on a Linux cluster. The symmetries are computed by a package written in GAP and python.

(8.8) In theorem B nothing is stated about $H_2^{-,+}$. Numerically one finds that $H_2^{-,+}$-symmetric minima are rotating Lagrange triangles, but (2.20) blow cannot be applied to this case.

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